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Controllability of impulsive semilinear pure integro-differential equations with delays and nonlocal conditions

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Abstract. In this paper, we study the exact and approximate controllability of a class of semilinear integro-differential control systems involving impulses, delays, and nonlocal conditions. In contrast to standard approaches, where the corresponding linear dynamics are transformed into an ordinary differential equation and memory effects are incorporated as perturbations, we preserve the original Volterra integro-differential structure of the system.

Our approach starts with a detailed analysis of the controllability properties of the associated linear integro-differential system. Based on this characterization, we then address the semilinear problem. Under appropriate assumptions on the nonlinear contributions, including those induced by delay-dependent terms, impulsive effects, and nonlocal conditions, we establish exact controllability by employing Rothe's fixed-point theorem and a suitable subadditivity framework.

Furthermore, by weakening the conditions imposed on the nonlinear terms, we obtain approximate controllability results through an alternative technique inspired by A.E. Bashirov et al., which allows us to avoid the use of fixed-point arguments.

1. Introduction

In this work, we address the exact and approximate controllability of a semilinear integro-differential control system incorporating impulses, delays, and nonlocal conditions. In contrast with standard treatments, where the associated linear dynamics are typically reduced to an ordinary differential

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equation and memory effects are incorporated as perturbations, we preserve the intrinsic Volterra integro-differential structure of the linear part.

The analysis is developed in two main stages. We begin by establishing a controllability characterization for the corresponding linear Volterra system (see Lemma 2). Building on this result, we then consider the semilinear problem. Assuming that the underlying linear system is controllable and that the nonlinear terms namely, the delay-dependent external force, the impulsive contributions, and the nonlocal component satisfy suitable sub-linear growth conditions, we prove exact controllability by means of Rothe's fixed-point theorem.

Subsequently, we derive approximate controllability under weaker assumptions. In particular, we impose a single growth condition on the delayed external force together with mild regularity requirements on the remaining nonlinear terms. The argument relies on a technique introduced by A.E. Bashirov et al. (see [2–4]), which allows us to obtain approximate controllability results without resorting to fixed-point methods. To the best of our knowledge, this approach provides a novel perspective and suggests new directions in the study of integro-differential systems with memory, impulses, and delay effects.

It is worth noting that, in many models with hereditary features, the associated linear control system takes the form of an ordinary differential equation. In such cases, the controllability operator depends explicitly on the memory kernel through the Volterra resolvent R, since the input-to-state mapping at time τ is given by

$$u \mapsto \int_0^\tau R(\tau, s)B(s)u(s)\, ds,$$

even when the control appears in the instantaneous form $B(t)u(t)$. This phenomenon is well documented in the context of partial differential equations, where systems such as heat or viscoelastic models with memory may exhibit either positive or negative controllability properties depending on the kernel structure and the actuation mechanism [1, 7, 9].

More generally, control problems for dynamical systems with memory arise in a wide range of applications, including heat conduction with hereditary effects, viscoelastic materials, population dynamics, and related fields. A significant portion of the literature on semilinear controllability relies on reducing the associated linear dynamics to an ordinary differential equation of the form

$$z'(t) = \mathfrak{A}(t)\, z(t) + \mathfrak{B}(t)\, u(t), \tag{1}$$

and subsequently treating delays, nonlocal conditions, or memory terms as nonlinear perturbations within this ODE framework; see, for instance, classical results on (approximate) controllability in Banach and Hilbert spaces

[11]. Within this setting, controllability of the linear system (1) is typically assumed, and the corresponding semilinear problem is handled under boundedness, Lipschitz-type, or growth conditions on the nonlinear terms.

In this paper, we consider the following semilinear control system involving impulses, delays, and a nonlocal condition

$$\begin{cases} z'(t) = C(t)z(t) + \int\limits_0^t \big[A(t,s)z(s) + B(t,s)u(s)\big]\, ds \\ \qquad\qquad + f(t, z_t, u(t)), \quad t \in (0,\tau], \\ z(s) + \mathcal{P}\big(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}\big)(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{S}_k\big(z(t_k), u(t_k)\big), \quad k = 1, 2, \dots, p, \end{cases} \tag{2}$$

where $0 < t_1 < t_2 < \cdots < t_p < \tau$ and $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$. The state variable satisfies $z(t) \in \mathbb{R}^n$, while the control function is given by $u(t) \in \mathbb{R}^m$. Moreover, the history segment $z_t \colon [-r, 0] \to \mathbb{R}^n$ is defined by $z_t(s) = z(t+s)$. The coefficient matrices $C(t) \in \mathbb{R}^{n\times n}$, $A(t,s) \in \mathbb{R}^{n\times n}$, and $B(t,s) \in \mathbb{R}^{n\times m}$ are assumed to be continuous. In contrast with approaches based on reducing the problem to an ordinary differential equation, we retain the Volterra-type structure at the linear level and analyze the controllability of a genuine integro-differential system, where both the state and the control act through kernels depending on two time variables. In particular, we consider the associated linear system

$$z'(t) = C(t)z(t) + \int_0^t [A(t,s)z(s) + B(t,s)u(s)]\, ds, \quad t \in (0,\tau]. \tag{3}$$

Assumptions for Exact Controllability:
The mappings

$$\begin{aligned} &\mathcal{P} \colon C_q := C\big([-r,0]; (\mathbb{R}^n)^q\big) \to C\big([-r,0]; \mathbb{R}^n\big), \quad \phi \in C\big([-r,0]; \mathbb{R}^n\big), \\ &f \colon [0,\tau] \times C\big([-r,0]; \mathbb{R}^n\big) \times \mathbb{R}^m \to \mathbb{R}^n, \\ &\mathcal{S}_k \in C\big(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n\big),\ k = 1, \dots, p, \end{aligned}$$

are assumed to satisfy the following sublinear growth conditions:

$$\begin{aligned} \|f(t,\phi,u)\|_{\mathbb{R}^n} &\le a_0 \|\phi(-r)\|_{\mathbb{R}^n}^{\varsigma_0} + b_0 \|u\|_{\mathbb{R}^m}^{\nu_0} + c_0, \\ &\qquad u \in \mathbb{R}^m,\ \phi \in C([-r,0]; \mathbb{R}^n),\ t \in [0,\tau], \end{aligned} \tag{4}$$

$$\begin{aligned} \|\mathcal{S}_k(z,u)\|_{\mathbb{R}^n} &\le a_k \|z\|_{\mathbb{R}^n}^{\varsigma_k} + b_k \|u\|_{\mathbb{R}^m}^{\nu_k} + c_k, \\ &\qquad u \in \mathbb{R}^m,\ z \in \mathbb{R}^n,\ k = 1, 2, \dots, p, \end{aligned} \tag{5}$$

$$\|\mathcal{P}(z)\|_C \le e\, \|z\|_{C_q}^{\eta}, \qquad z \in C_q, \tag{6}$$

$$\|\mathcal{P}(z) - \mathcal{P}(w)\|_C \le K\, \|z - w\|_{C_q}, \qquad z, w \in C_q, \tag{7}$$

where the exponents satisfy $0 < \eta < 1$, $0 < \varsigma_k < 1$, and $0 < \nu_k < 1$ for all $k = 0, 1, \ldots, p$. In addition, the impulsive conditions are understood in the sense that

$$z(t_k^+) = \lim_{t \to t_k^+} z(t), \qquad z(t_k) = z(t_k^-) = \lim_{t \to t_k^-} z(t).$$

Assumptions for Approximate Controllability:
The main assumption is that the nonlinear mappings f and $\mathcal{S}_k$ possess sufficient regularity so that, for each control u, system (2) admits a unique solution in the sense of [5, 12]. Moreover, they are required to satisfy the estimate

$$(8) \qquad |f(t, \phi, u)| \leq \rho(\|\phi(-r)\|), \quad u \in \mathbb{R}^m, \quad \phi \in C([-r, 0]; \mathbb{R}^n),$$

where $\rho : \mathbb{R}_+ \to [0, \infty)$ is a continuous function. In particular, one may take $\rho(z) = a\|z\|^\nu + b$, with $\nu \geq 1$.

To formulate the problem, we introduce the natural Banach space

$$\begin{aligned}&\mathscr{PW}(J_r) = \mathscr{PW}(J_r; \mathbb{R}^n)\\ &\quad =\big\{z : J_r \to \mathbb{R}^n : z \in C(J';\ \mathbb{R}^n),\ \exists\ z(t_k^+),\ z(t_k^-) \text{ and } z(t_k) = z(t_k^-)\big\},\end{aligned}$$

where $J_r = [-r, \tau]$, $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$, and $J = [0, \tau]$, endowed with the norm

$$\|z\|_0 = \sup_{t \in J_r} \|z(t)\|_{\mathbb{R}^n}.$$

We also denote

$$\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \prod_{k=1}^{q} \mathbb{R}^n = (\mathbb{R}^n)^q,$$

equipped with the norm

$$\|z\|_q^n = \sum_{i=1}^{q} \|z_i\|_{\mathbb{R}^n}, \quad z = (z_1, z_2, \ldots, z_q)^\mathsf{T} \in (\mathbb{R}^n)^q.$$

Accordingly, the norm in the space $C([-r, 0]; (\mathbb{R}^n)^q)$ is defined by

$$\|z\|_{C_q} = \sup_{t \in [-r,0]} \|z(t)\|_q^n = \sup_{t \in [-r,0]} \left(\sum_{i=1}^{q} \|z_i(t)\|_{\mathbb{R}^n} \right), \quad \forall z \in C([-r, 0]; (\mathbb{R}^n)^q).$$

Furthermore, we consider the product Banach space

$$\mathscr{PW}(J_r) \times C(J) = \mathscr{PW}(J_r; \mathbb{R}^n) \times C(J; \mathbb{R}^m),$$

endowed with the norm

$$|||(z, u)||| = \|z\|_0 + \|u\|_0.$$

In $\mathbb{R}^n \times \mathbb{R}^m$, we use the norm

$$\|(z, u)\|_1 = \|z\|_{\mathbb{R}^n} + \|u\|_{\mathbb{R}^m} = \|z\| + \|u\|, \quad \forall (z, u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Finally, for each $(z,u) \in \mathscr{PW}(J_r) \times C(J)$, we define

$$\|f(\cdot,u,z)\|_0 = \sup_{t\in[0,\tau]} \|f(t,u(t),z(t))\|_{\mathbb{R}^n}.$$

Definition 1 (**Exact Controllability**)**.** The system (2) is said to be exactly controllable on $[0,\tau]$ if for every $\phi \in C([-r,0];\mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$, there exists $u \in L_2([0,\tau];\mathbb{R}^m)$ such that the solution $z(t)$ of (2) corresponding to u verifies:

$$z(0) + \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) = \phi(0) \quad \text{and} \quad z(\tau) = z_1.$$

Definition 2 (**Approximate Controllability**)**.** The system (2) is said to be approximately controllable on the interval $[0,\tau]$ if, for every $\phi \in C([-r,0];\mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$, and $\epsilon > 0$, there exists a control $u \in L_2([0,\tau];\mathbb{R}^m)$ such that the corresponding solution $z(t)$ of (2) satisfies

$$z(0) + \mathcal{P}(z_{\tau_1}, \ldots, z_{\tau_q})(0) = \phi(0), \quad \text{and} \quad \left\|z(\tau) - z^1\right\|_{\mathbb{R}^n} < \epsilon.$$

Theorem 1 (Rothe's Fixed Point Theorem, [10, 14])**.** *Let X be a Banach space and let $\mathcal{D} \subset X$ be a closed and convex set whose interior contains the origin. Suppose that $\mathcal{T} : \mathcal{D} \to X$ is a continuous operator such that $\mathcal{T}(\mathcal{D})$ is relatively compact in X and $\mathcal{T}(\partial\mathcal{D}) \subset \mathcal{D}$. Then there exists $x_0 \in \mathcal{D}$ such that $\mathcal{T}(x_0) = x_0$.*

2. Controllability of linear systems

In this section, we establish new characterizations of the controllability of the linear system (3) in the absence of impulses, delays, and nonlocal terms. We begin with a lemma that characterizes the exact controllability of (3); this result is a central novelty of the paper.

Lemma 1. *System* (3) *fails to be controllable if and only if there exists a nonzero function $\varphi \in C([0,\tau];\mathbb{R}^n)$ satisfying*

$$\varphi'(t) = -C^*(t)\varphi(t) - \int_t^\tau A^*(s,t)\varphi(s)ds, \quad t \in [0,\tau], \tag{9}$$

and

$$\int_t^\tau B^*(s,t)\varphi(s)ds = 0, \quad t \in [0,\tau]. \tag{10}$$

Proof. Suppose that system (3) is not controllable. Define

$$D = \left\{z(\tau) : z \text{ is a solution of } (3)\right\}.$$

Then D is a linear subspace of $\mathbb{R}^n$ and $D \neq \mathbb{R}^n$. Hence, there exists a nonzero vector $a \in \mathbb{R}^n$ such that

$$\langle a, z(\tau)\rangle = 0, \qquad \forall z(\tau) \in D.$$

Consider the function φ defined as the solution of

$$
(11) \qquad \begin{cases} \varphi'(t) = -C^*(t)\varphi(t) - \int_t^\tau A^*(s,t)\varphi(s)ds, & t \in [0,\tau], \\ \varphi(\tau) = a. \end{cases}
$$

Let $u \in L_\infty^m[0,\tau]$ be arbitrary and let $z(t)$ denote the corresponding solution of (3). We compute

$$
\begin{aligned}
\int_0^\tau \langle z(t), \varphi'(t)\rangle\, dt &= \int_0^\tau \left\langle z(t), -\int_t^\tau A^*(s,t)\varphi(s)\, ds \right\rangle dt \\
&\quad - \int_0^\tau \langle z(t), C^*(t)\varphi(t)\rangle dt \\
&= -\int_0^\tau \int_t^\tau \langle z(t), A^*(s,t)\varphi(s)\rangle ds\, dt \\
&\quad - \int_0^\tau \langle C(t)z(t), \varphi(t)\rangle dt \\
&= -\int_0^\tau \left\langle \int_0^t A(t,s)z(s)\, ds, \varphi(t) \right\rangle dt \\
&\quad - \int_0^\tau \langle C(t)z(t), \varphi(t)\rangle dt \\
&= -\int_0^\tau \left\langle (C(t)z(t) + \int_0^t A(t,s)z(s)\, ds, \varphi(t) \right\rangle dt.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\int_0^\tau \langle z'(t), \varphi(t)\rangle\, dt &= \int_0^\tau \left\langle C(t)z(t) + \int_0^t A(t,s)z(s)\, ds, \varphi(t) \right\rangle dt \\
&\quad + \int_0^\tau \left\langle \int_0^t B(t,s)u(s)\, ds, \varphi(t) \right\rangle dt.
\end{aligned}
$$

Applying integration by parts and using $\varphi(\tau) = a$, we obtain

$$
\begin{aligned}
\int_0^\tau \langle z'(t), \varphi(t)\rangle\, dt &= \langle \varphi(\tau), z(\tau)\rangle - \langle \varphi(0), z(0)\rangle - \int_0^\tau \langle \varphi'(t), z(t)\rangle\, dt \\
&= \int_0^\tau \left\langle C(t)z(t) + \int_0^t A(t,s)z(s)\, ds, \varphi(t) \right\rangle dt.
\end{aligned}
$$

Combining the previous expressions yields

$$
\begin{aligned}
0 &= \int_0^\tau \left\langle \int_0^t B(t,s)u(s)\, ds, \varphi(t) \right\rangle dt \\
&= \int_0^\tau \left\langle \int_t^\tau B^*(s,t)\varphi(s)\, ds, u(t) \right\rangle dt,
\end{aligned}
$$

for all $u \in L_\infty^m[0,\tau]$. Therefore,

$$
\int_t^\tau B^*(s,t)\varphi(s)\, ds = 0, \qquad t \in [0,\tau].
$$

Conversely, assume that there exists a function φ with $\varphi(T) \neq 0$ satisfying (9) and (10). Let z be any solution of (3). Repeating the previous arguments and applying integration by parts, we obtain

$$\begin{aligned}
0 &= \int_0^\tau \Big\langle \int_t^\tau B^*(s,t)\varphi(s)\,ds, u(t)\Big\rangle dt \\
&= \int_0^\tau \Big\langle \int_0^t B(t,s)u(s)\,ds, \varphi(t)\Big\rangle dt \\
&= \int_0^\tau \langle z'(t), \varphi(t)\rangle\,dt - \int_0^\tau \Big\langle C(t)z(t) + \int_0^t A(t,s)z(s)\,ds, \varphi(t)\Big\rangle dt \\
&= \varphi(\tau)z(\tau) - \varphi(0)z(0) = \varphi(\tau)z(\tau).
\end{aligned}$$

This shows that D admits a nonzero orthogonal vector $\varphi(\tau) \neq 0$. Hence, D is not dense in $\mathbb{R}^n$, that is, $D \neq \mathbb{R}^n$. Consequently, system (3) is not controllable. □

2.1. **A characterization of the controllability of (3) in terms of linear operators.** In this subsection, we provide an operator-theoretic characterization of the controllability of the linear integro-differential system (3). The goal is to express the controllability property in terms of suitable linear mappings acting on the control space.

To this end, consider an arbitrary initial state $z_0 \in \mathbb{R}^n$ and a control function $u \in L_2([0,\tau];\mathbb{R}^m)$. The corresponding initial value problem

$$\text{(12)} \qquad \begin{cases} z'(t) = C(t)z(t) + \displaystyle\int_0^t [A(t,s)z(s) + B(t,s)u(s)]\,ds, & t \in [0,\tau], \\ z(0) = z_0, \end{cases}$$

admits a unique solution, which can be written in the integral form

$$\text{(13)} \qquad z(t) = \mathcal{U}(t,0)z_0 + \int_0^t \int_0^s \mathcal{U}(t,s)B(s,l)u(l)dlds, \qquad t \in [0,\tau],$$

where $\mathcal{U}(t,s)$ denotes the principal matrix solution associated with the system. The matrix $\mathcal{U}(t,s)$ is defined as the solution of

$$\frac{\partial}{\partial t}\mathcal{U}(t,s) = C(t)\mathcal{U}(t,s) + \int_s^t A(t,r)\mathcal{U}(r,s)\,dr, \qquad \mathcal{U}(s,s) = I,$$

and satisfies the cocycle property $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$ and the bound

$$\text{(14)} \qquad \|\mathcal{U}(t,s)\| \leq Me^{\omega(t-s)}, \quad \omega > 0.$$

(see [6, 15]). From now on, we shall denote

$$\text{(15)} \qquad \|B\| = \max\big\{\|B(t,s)\| : t,s \in [0,\tau]\big\}.$$

We now introduce the controllability operator associated with the system (3).

Definition 3. For $\tau > 0$, the controllability operator $\mathscr{G} : L_2([0,\tau];\mathbb{R}^m) \to \mathbb{R}^n$ is defined by

$$\mathscr{G}u = \int_0^\tau \int_0^s \mathcal{U}(\tau,s)B(s,l)u(l)\,dl\,ds \\ = \int_0^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l)u(l)\,ds\,dl. \tag{16}$$

Proposition 1. *The adjoint operator $\mathscr{G}^* : \mathbb{R}^n \to L_2([0,\tau];\mathbb{R}^m)$ is given by*

$$(\mathscr{G}^*\xi)(t) = \int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\xi\,ds, \quad t \in [0,\tau],\ \xi \in \mathbb{R}^n, \tag{17}$$

and the corresponding controllability Gramian $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^n$ takes the form

$$\mathcal{W}\xi = \mathscr{G}\mathscr{G}^*\xi = \int_0^\tau \int_l^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l)B^*(k,l)\mathcal{U}^*(\tau,k)\xi\,dk\,ds\,dl. \tag{18}$$

Proof. We derive $\mathscr{G}^*$ from the identity

$$\langle \mathscr{G}u, \xi\rangle = \langle u, \mathscr{G}^*\xi\rangle, \qquad \forall \xi \in \mathbb{R}^n,\ \forall u \in L_2([0,\tau];\mathbb{R}^m).$$

Indeed,

$$\begin{aligned} \langle \mathscr{G}u, \xi\rangle &= \left\langle \int_0^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l)u(l)\,ds\,dl, \xi \right\rangle \\ &= \int_0^\tau \left\langle \int_l^\tau \mathcal{U}(\tau,s)B(s,l)u(l)\,ds, \xi \right\rangle dl \\ &= \int_0^\tau \left\langle u(l), \int_l^\tau B^*(s,l)\mathcal{U}^*(\tau,s)\xi\,ds \right\rangle dl. \end{aligned}$$

This proves the expression for $\mathscr{G}^*$. Next, we compute $\mathscr{G}\mathscr{G}^*$:

$$\begin{aligned} \mathscr{G}\mathscr{G}^*\xi &= \int_0^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l)(\mathscr{G}^*\xi)(l)\,ds\,dl \\ &= \int_0^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l) \int_l^\tau B^*(k,l)\mathcal{U}^*(\tau,k)\xi\,dk\,ds\,dl \\ &= \int_0^\tau \int_l^\tau \int_l^\tau \mathcal{U}(\tau,s)B(s,l)B^*(k,l)\mathcal{U}^*(\tau,k)\xi\,dk\,ds\,dl. \end{aligned}$$

□

Proposition 2. *System* (3) *is controllable on* $[0,\tau]$ *if and only if* $\mathrm{Range}(\mathscr{G}) = \mathbb{R}^n$.

By a classical result [8, p. 55], we obtain the following equivalent characterizations.

Lemma 2. *The following statements are equivalent:*

a) $\mathrm{Range}(\mathscr{G}) = \mathbb{R}^n$.
b) $\ker(\mathscr{G}^*) = \{0\}$.
c) *There exists $\gamma > 0$ such that $\langle \mathcal{W}z, z\rangle > \gamma\|z\|^2$ for all $z \neq 0$.*

d) *The operator* $\mathcal{W}$ *is invertible in* $\mathbb{R}^n$.

e) $\displaystyle\int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\xi\,ds = 0\ \forall t\in[0,\tau]\ \Rightarrow\ \xi = 0.$

Accordingly, the operator $\Upsilon : \mathbb{R}^n \to L_2([0,\tau];\mathbb{R}^m)$ defined by

$$(\Upsilon z)(t) = \int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\mathcal{W}^{-1}z\,ds = (\mathscr{G}^*(\mathscr{G}\mathscr{G}^*)^{-1}z)(t), \tag{19}$$

is called the steering operator and it satisfies

$$\mathscr{G}\Upsilon = I. \tag{20}$$

Moreover,

$$\|\mathcal{W}^{-1}z\| \le \gamma^{-1}\|z\|, \qquad z\in\mathbb{R}^n, \tag{21}$$

and a control that steers the system (3) from z_0 to z_1 at time T is given by

$$\begin{aligned} u(t) &= \int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\,ds\,\mathcal{W}^{-1}(z_1 - \mathcal{U}(\tau,0)z_0) \\ &= \Upsilon(z_1 - \mathcal{U}(\tau,0)z_0)(t), \quad t\in[0,\tau]. \end{aligned} \tag{22}$$

Lemma 3 (cf. [13]). *Let S be a dense subspace of $L_2(J_r)$. Then the system (3) is controllable by means of controls in $L_2(J_r)$ if and only if it is controllable using controls taken from S. In other words,*

$$\mathrm{Range}(\mathscr{G}) = \mathbb{R}^n \quad\Longleftrightarrow\quad \mathrm{Range}(\mathscr{G}|_S) = \mathbb{R}^n.$$

Remark 1. An immediate consequence of the previous result is that controllability may be realized through controls belonging to dense subspaces of $L_2(J_r)$, for instance,

$$C([0,\tau];\mathbb{R}^m), \quad C^\infty([0,\tau];\mathbb{R}^m), \quad L_\infty([0,\tau];\mathbb{R}^m).$$

Moreover, the operators $\mathscr{G}$, $\mathcal{W}$, and Υ admit natural extensions to spaces such as $C(J)$, preserving their structural form.

3. Exact controllability of the semilinear system (2)

In this section, we address the exact controllability of the semilinear integro-differential system (2), which incorporates impulses, delays, and nonlocal conditions. The strategy consists in reducing the nonlinear controllability problem to a fixed-point formulation built upon the controllability properties of the associated linear system.

More precisely, for a given initial history $\phi\in C([-r,0];\mathbb{R}^n)$ and a control function $u\in C(J)$, it follows from [5, 12] that the problem

$$\begin{cases} z'(t) = C(t)z(t) + \displaystyle\int_0^t [A(t,s)z(s) + B(t,s)u(s)]\,ds \\ \qquad\qquad + f(t, z_t, u(t)), \quad t\in(0,\tau], \\ z(s) + \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(s) = \phi(s), \quad s\in[-r,0], \\ z(t_k^+) = z(t_k^-) + \mathcal{S}_k(z(t_k), u(t_k)), \quad k = 1,\dots,p, \end{cases} \tag{23}$$

admits a unique mild solution, which can be represented as

$$
\left\{
\begin{aligned}
z^u(t) = {}& \mathcal{U}(t,0)\big[\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\big] \\
& + \int_0^t \int_0^s \mathcal{U}(t,s)B(s,l)u(l)dlds \\
& + \int_0^t \mathcal{U}(t,s)f(s, z_s^u, u(s))ds \\
& + \sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{S}_k(z(t_k), u(t_k)), \quad t \in [0,\tau], \\
z(t) = {}& \phi(t) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t), \quad t \in [-r, 0].
\end{aligned}
\right.
\tag{24}
$$

This representation highlights the combined effect of the linear dynamics, the distributed memory term, the nonlinear perturbation, and the impulsive contributions. It also provides the appropriate framework to construct a control that steers the system to a prescribed final state. We are now in a position to establish the main result of this section.

Theorem 2. *Suppose that assumptions* (4)–(7) *hold and that the associated linear system* (3) *is controllable on* $[0,\tau]$. *Then the semilinear system* (2) *is exactly controllable on* $[0,\tau]$.

More precisely, for every initial function $\phi \in C([-r,0];\mathbb{R}^n)$ *and every target state* $z^1 \in \mathbb{R}^n$, *there exists a control* $u \in C(J_r)$ *such that the corresponding solution* $z^u(\cdot)$ *satisfies*

$$
\begin{aligned}
z^1 &= \mathcal{U}(\tau,0)\big[\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\big] \\
&\quad + \int_0^\tau \int_0^s \mathcal{U}(\tau,s)\, B(s,l)u(l)\, dl\, ds + \int_0^\tau \mathcal{U}(\tau,s)\, f\big(s, z_s^u, u(s)\big)\, ds \\
&\quad + \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)\, \mathcal{S}_k\big(z(t_k), u(t_k)\big).
\end{aligned}
$$

Furthermore, the control can be explicitly constructed as

$$
u(t) = \int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\, \mathcal{W}^{-1}\mathfrak{L}(z,u)\, ds = (\Upsilon\, \mathfrak{L}(z,u))(t), \qquad t \in [0,\tau],
$$

where the functional $\mathfrak{L}(z,u)$ *is defined by*

$$
\begin{aligned}
\mathfrak{L}(z,u) = z^1 &- \mathcal{U}(\tau,0)\big[\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\big] \\
&- \int_0^\tau \mathcal{U}(\tau,s)\, f\big(s, z_s, u(s)\big)\, ds \\
&- \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)\, \mathcal{S}_k\big(z(t_k), u(t_k)\big).
\end{aligned}
$$

Proof. To prove the theorem, we introduce the operator

$$
\Phi : \mathscr{PW}(J_r) \times C(J) \longrightarrow \mathscr{PW}(J_r) \times C(J),
$$

defined by

$$
\Phi(z,u) = \big(\Phi_1(z,u), \Phi_2(z,u)\big) =: (y,v).
$$

Here, the components Φ_1 and Φ_2 are given by

$$\Phi_1 : \mathscr{PW}(J_r) \times C(J) \longrightarrow \mathscr{PW}(J_r), \qquad \Phi_2 : \mathscr{PW}(J_r) \times C(J) \longrightarrow C(J).$$

For each $t \in [0, \tau]$, we define

$$\begin{aligned} y(t) &= \Phi_1(z,u)(t) \\ &= \mathcal{U}(t,0)\big\{\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\big\} \\ &\quad + \int_0^t \int_0^s \mathcal{U}(t,s)B(s,l)\Upsilon\mathfrak{L}(z,u)(l)\, dl\, ds \\ &\quad + \int_0^t \mathcal{U}(t,s)f(s,z_s,u(s))\, ds + \sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{S}_k(z(t_k),u(t_k)), \end{aligned} \tag{25}$$

and

$$\begin{aligned} v(t) &= \Phi_2(z,u)(t) = (\Upsilon\mathfrak{L}(z,u))(t) \\ &= \int_t^\tau B^*(s,t)\mathcal{U}^*(\tau,s)\, ds\, \mathcal{W}^{-1}\mathfrak{L}(z,u), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathfrak{L}(z,u) &= z_1 - \mathcal{U}(\tau,0)\big[\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\big] \\ &\quad - \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s,u(s))\, ds \\ &\quad - \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)\mathcal{S}_k(z(t_k),u(t_k)). \end{aligned} \tag{27}$$

With this construction, the controllability of the semilinear system (2) is reduced to a fixed-point problem. More precisely, for every prescribed initial function $\phi \in C([-r,0];\mathbb{R}^n)$ and every target state $z_1 \in \mathbb{R}^n$, system (2) is controllable if and only if the operator Φ, defined by (25)–(27), admits a fixed point; that is,

$$\exists\,(z,u) \in \mathrm{Dom}(\Phi) \qquad \text{such that} \qquad \Phi(z,u) = (z,u).$$

Following the same line of argument used in [13], one proves that Φ is continuous and compact. In addition, Φ satisfies a sublinear estimate of the form

$$\begin{aligned} \|\Phi(z,u)\| &\le L_1 + L_2\big\{e\|z\|^\eta + a_0\|z\|^{\varsigma_0} + b_0\|u\|^{\nu_0} + c_0\big\} \\ &\quad + L_3 \sum_{0<t_k<T} \big\{a_k\|z\|^{\varsigma_k} + b_k\|u\|^{\nu_k} + c_k\big\}, \end{aligned}$$

where L_1, L_2, and L_3 are positive constants depending only on M, $\|B\|$, $\|\mathcal{W}^{-1}\|$, ω, and K. Dividing both sides by

$$\|(z,u)\| = \|z\|_0 + \|u\|_0,$$

we obtain

$$\lim_{\|(z,u)\|\to\infty} \frac{\|\Phi(z,u)\|}{\|(z,u)\|} = 0.$$

Hence, for any fixed number $0 < \lambda < 1$, there exists $R > 0$ sufficiently large such that

$$\|\Phi(z,u)\| \leq \lambda \,\|(z,u)\| \qquad \text{whenever } \|(z,u)\| \geq R.$$

Therefore,

$$\Phi(\partial B(0,R)) \subset B(0,R).$$

Since Φ is continuous and compact, Rothe's fixed point theorem (Theorem 1) ensures the existence of a point

$$(z,u) \in B(0,R) \subset \mathscr{PW}(J_r) \times C(J)$$

such that

$$\Phi(z,u) = (z,u).$$

This proves that system (2) is controllable on $[0,\tau]$. Moreover, the control is given by

$$u = \Upsilon\mathfrak{L}(z,u) = B^*(\cdot)\,\mathcal{U}^*(\tau,\cdot)\,\mathcal{W}^{-1}\mathfrak{L}(z,u),$$

and, for the prescribed initial datum $\phi \in C([-r,0];\mathbb{R}^n)$ and target state $z^1 \in \mathbb{R}^n$, the corresponding solution $z(t) = z(t,\phi,u)$ of (2) satisfies

$$\begin{aligned} z^1 = {}& \mathcal{U}(\tau,0)\big[\phi(0) - \mathcal{P}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\big] \\ &+ \int_0^\tau\!\int_0^s \mathcal{U}(\tau,s)B(s,l)u(l)\,dl\,ds \\ &+ \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s^u,u(s))\,ds \\ &+ \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)\mathcal{S}_k(z(t_k),u(t_k)). \end{aligned}$$

This completes the proof. □

4. Approximate controllability of the semilinear system (2)

In this section, we establish the approximate controllability of system (2). The argument relies on combining the exact controllability of a truncated linear system with suitable estimates on the nonlinear terms.

We assume that the nonlinear functions satisfy condition (8) and that the following linear system

$$\left\{ \begin{aligned} & y'(t) = C(t)y(t) + \int_{\tau-\delta}^t [A(t,s)y(s) + B(t,s)u(s)]\,ds, \\ & y(\tau-\delta) = z_0, \end{aligned} \right. \tag{28}$$

is exactly controllable on every interval $[\tau-\delta,\tau]$, for all $0 < \delta < \tau$. For any initial state $z_0 \in \mathbb{R}^n$ and control $u \in L^2([\tau-\delta,\tau];\mathbb{R}^m)$, system (28) admits a unique solution of the form

$$y(t) = \mathcal{U}(t,\tau-\delta)z_0 + \int_{\tau-\delta}^t \int_{\tau-\delta}^s \mathcal{U}(\tau,s)B(s,l)u(l)\,dl\,ds, \quad t \in [\tau-\delta,\tau]. \tag{29}$$

In particular, given $z_0, z^1 \in \mathbb{R}^n$, there exists a control v^δ such that the corresponding solution satisfies

$$(30) \qquad y^\delta(\tau) = \mathcal{U}(\tau, \tau-\delta)z_0 + \int_{\tau-\delta}^{\tau}\int_{\tau-\delta}^{s} \mathcal{U}(\tau, s)B(s,l)v^\delta(l)\, dl\, ds = z^1.$$

We now turn to the semilinear system (2). For every $\phi \in C([-r, 0]; \mathbb{R}^n)$ and $u \in L^2([0, \tau]; \mathbb{R}^m)$, it is known from [5, 12] that there exists a unique solution $z \in \mathscr{PW}(J_r)$ given by the variation-of-constants formula.

Theorem 3. *Assume that the nonlinear terms f and $\mathcal{S}_k$ satisfy the regularity condition* (8) *and that the linear system* (28) *is exactly controllable on every interval $[\tau-\delta, \tau]$, $0 < \delta < \tau$. Then the semilinear system* (2) *is approximately controllable on* $[0, \tau]$.

Proof. Let $\phi \in C([-r, 0]; \mathbb{R}^n)$, a target state $z^1 \in \mathbb{R}^n$, and $\epsilon > 0$ be given. Our goal is to construct a control $u^\delta \in L^2([0, \tau]; \mathbb{R}^m)$ such that the corresponding solution satisfies

$$\|z^\delta(\tau) - z^1\| < \epsilon.$$

We proceed as follows. First, fix an arbitrary control $u \in L^2([0, \tau]; \mathbb{R}^m)$ and denote by $z(t) = z(t, 0, \phi, u)$ the corresponding solution of (2). Define

$$K = \max\{\rho(\|z(t)\|) : t \in [0, \tau]\}.$$

Next, choose $\delta > 0$ sufficiently small such that

$$0 < \delta < \min\left\{\frac{\epsilon}{MK}, \tau - t_p, r\right\}.$$

We now construct a new control u^δ by modifying u only on the final interval $[\tau-\delta, \tau]$:

$$u^\delta(t) = \begin{cases} u(t), & 0 \le t \le \tau-\delta, \\ v^\delta(t), & \tau-\delta < t \le \tau, \end{cases}$$

where v^δ is a control that steers the linear system (28) from the state $z(\tau-\delta)$ to the target z^1. Since $\delta < \tau - t_p$, it follows that $\tau - \delta > t_p$, and therefore no impulses occur in the interval $[\tau-\delta, \tau]$. Using the cocycle property of $\mathcal{U}$, we can write the solution of (2) at time τ as

$$\begin{aligned} z^\delta(\tau) = {} & \mathcal{U}(\tau, \tau-\delta)z(\tau-\delta) \\ & + \int_{\tau-\delta}^{\tau}\int_{\tau-\delta}^{s} \mathcal{U}(\tau, s)B(s,\tau)v^\delta(\tau)\, d\tau\, ds \\ & + \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau, s)f(s, z_s^\delta, v^\delta(s))\, ds. \end{aligned}$$

On the other hand, the solution y^δ of the linear system (28) satisfies

$$y^\delta(\tau) = \mathcal{U}(\tau, \tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau}\int_{\tau-\delta}^{s} \mathcal{U}(\tau, s)B(s,\tau)v^\delta(\tau)\, d\tau\, ds,$$

and by construction,

$$y^{\delta}(\tau) = z^1.$$

Subtracting both expressions, we obtain

$$\|z^{\delta}(\tau) - z^1\| \leq \int_{\tau-\delta}^{\tau} \|\mathcal{U}(\tau, s)\| \cdot \|f(s, z_s^{\delta}, u^{\delta}(s))\| \, ds.$$

Since $0 < \delta < r$ and $s \in [\tau - \delta, \tau]$, we have $s - r < \tau - \delta$, which implies

$$z^{\delta}(s - r) = z(s - r).$$

Therefore, using condition (8),

$$\|f(s, z_s^{\delta}, u^{\delta}(s))\| \leq \rho(\|z^{\delta}(s - r)\|) = \rho(\|z(s - r)\|) \leq K.$$

It follows that

$$\begin{aligned} \|z^{\delta}(\tau) - z^1\| &\leq \int_{\tau-\delta}^{\tau} M e^{\omega(\tau - s)} K \, ds \\ &\leq MK \int_{\tau-\delta}^{\tau} ds = MK\,\delta < \epsilon. \end{aligned}$$

This proves that the system is approximately controllable on $[0, \tau]$. □

5. Final remarks

In this work, we have shown that preserving a genuinely integro-differential (Volterra-type) structure in the linear part is not only natural for systems with memory, but also particularly effective for the analysis of controllability properties in semilinear settings involving impulses, delays, and nonlocal conditions. Rather than reducing the model to an equivalent ordinary differential framework, our approach retains the hereditary dynamics and exploits their intrinsic structure.

More precisely, we have obtained three main contributions. First, we provide a characterization of the exact controllability of the associated linear system (3) in terms of an adjoint integro-differential equation (Lemma 2). This characterization is complemented by an operator-theoretic formulation through the controllability map, the explicit computation of its adjoint, and the introduction of the corresponding controllability Gramian, which plays a central role in the analysis.

Second, under the sublinearity assumptions (4)–(7), together with the controllability of the linear system, we establish the exact controllability of the semilinear system (2). The proof relies on a fixed-point framework based on Rothe's theorem, where the nonlinear contributions, impulses, and nonlocal terms are handled in a unified manner within an appropriate functional setting.

Third, by relaxing the structural assumptions on the nonlinearities and imposing only the growth condition (8) on the delayed external force, we prove approximate controllability of (2). The argument is based on a localization technique near the terminal time, combined with a Bashirov-type selection method, which avoids the use of fixed-point arguments and highlights the robustness of the controllability property under weaker hypotheses.

These results indicate that integro-differential models with memory effects admit a rich controllability theory, comparable in scope to that of classical evolution equations, while capturing more realistic dynamics.

As a natural continuation of this work, we plan to investigate optimal control problems associated with system (2). In particular, an interesting direction is the derivation of a Pontryagin-type maximum principle for integro-differential systems with impulses, delays, and nonlocal conditions, including the treatment of mixed cost functionals and constraints on both the state and the control variables.

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